A HYPERFINITE INEQUALITY FOR FREE ENTROPY DIMENSION

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ABSTRACT. If X, Y, Z are finite sets of selfadjoint elements in a tracial von Neumann algebra and X generates a hyperfinite von Neumann algebra, then $\delta_0(X \cup Y \cup Z) \leq \delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(X)$. We draw several corollaries from this inequality.

In [11] Voiculescu describes the role of entropy in free probability. He discusses several problems in the area, one of which is the free entropy dimension problem. Free entropy dimension ([8], [9]) associates to an n-tuple of selfadjoint operators, $X = \{x_1, \ldots, x_n\}$, in a tracial von Neumann algebra M a number $\delta_0(X)$ called the (modified) free entropy dimension of X. $\delta_0(X)$ is an asymptotic Minkowski or packing dimension of sets of n-tuples of matrices which model the behavior of X. The free entropy dimension problem simply asks whether $\delta_0(X) = \delta_0(Y)$ for any other m-tuple of selfadjoint elements Y satisfying $Y'' = X''$. It is known from [10] that δ_0 is an algebraic invariant, i.e., $\delta_0(X) = \delta_0(Y)$ if X and Y generate the same algebra.

The origin of this remark started with two extremely special and highly tractable cases of this problem, the first being: if X, Y and Z are finite sets of selfadjoint elements in M such that $X'' = Z''$ is hyperfinite, then is it true that

$$\delta_0(X \cup Y) = \delta_0(Y \cup Z)?$$

The second problem concerns invariance of δ_0 over the center: if Y is an arbitrary set of selfadjoint elements in M and y is any element in the center of Y'', then is it true that

$$\delta_0(Y \cup \{y\}) = \delta_0(Y)?$$

Both questions have affirmative answers and follow from a kind of hyperfinite inequality for δ_0 : If $X, Y, Z,$ are sets of selfadjoint elements in M and X generates a hyperfinite von Neumann algebra, then

$$\delta_0(X \cup Y \cup Z) \leq \delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(X).$$

Related inequalities of this nature can be found in Gaboriau's work on the cost of equivalence relations [3]. The proof of the microstates inequality above is an application of the work in [5] paired with the packing formulation of δ_0 in [6].

This remark has four sections. The first is a short list of assumptions. Motivated by the recent work of Belinschi and Bercovici ([1]), the second presents a slightly simpler formulation of δ_0 where the operator cutoff constants are removed. The third section presents the hyperfinite inequality. The fourth and last section presents several corollaries, two of which answer the special invariance questions posed above. It is also shown that δ_0 shares a certain property of the Connes-Shlyakhtenko dimension, $\triangle$. In [2] Connes and Shlyakhtenko show that if F is a finite set of selfadjoints in M with F'' having diffuse center, then $\triangle(F) = 1$. We show that δ_0 satisfies the same property provided that F is assumed to have finite dimensional approximants.

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1. Preliminaries

Throughout suppose M is a von Neumann algebra with a normal, tracial state φ. For any $n \in \mathbb{N}, |\cdot|_2$ denotes the norm on $(M_k^{sa}(\mathbb{C}))^n$ given by $|(x_1, \ldots, x_n)|_2 = (\sum_{j=1}^n tr_k(x_j^2))^{\frac{1}{2}}$ where tr_k is the tracial state on the $k \times k$ complex matrices, and $|\cdot|_\infty$ denotes the operator norm. U_k denotes the $k \times k$ unitary matrices. For any $k, n \in \mathbb{N}, u \in U_k$ and $x = (x_1, \ldots, x_n) \in (M_k^{sa}(\mathbb{C}))^n$, define $uxu^* = (ux_1u^*, \ldots, ux_nu^*)$. We will maintain the notation introduced in [6], [8], and [9]. If $F = \{a_1, \ldots, a_n\}$ is a finite set of selfadjoint elements in M, we abbreviate $\Gamma_R(a_1, \ldots, a_n; m, k, \gamma)$ by $\Gamma_R(F; m, k, \gamma)$ and in a similar way we write the associated microstate sets and quantities introduced in [6], [8], and [9]: $\delta_0(F), \mathbb{P}_\epsilon(F), \mathbb{K}_\epsilon(F)$. Also, if $G = \{b_1, \ldots, b_p\}$ is another finite set of selfadjoint elements in M, then we denote by $\Gamma_R(F \cup G; m, k, \gamma)$ the set $\Gamma_R(a_1, \ldots, a_n, b_1, \ldots, b_p; m, k, \gamma)$ and write all the associated microstate quantities $\delta_0(F \cup G), \mathbb{P}_\epsilon(F \cup G)$, and $\mathbb{K}_\epsilon(F \cup G)$ with respect to $\Gamma_R(F \cup G; m, k, \gamma)$. Finally, $\Gamma(F; m, k, \gamma)$ will denote the set of all $k \times k$ microstates (no restrictions on the operator norms) with degree of approximation (m, γ)

2. Cutoff Constants

Recall that in [1] Belinschi and Bercovici have lifted the operator norm cutoff constants in the definition of χ. In other words, if $\chi(X)$ is the normal definition as conceived of by Voiculescu, and $\chi_\infty(X)$ is the quantity obtained by replacing the microstate spaces $\Gamma_R(X; m, k, \gamma)$ with $\Gamma(X; m, k, \gamma)$, then Belinschi and Bercovici showed that one always has

$$\chi(X) = \chi_\infty(X).$$

We want to show the same thing for the packing formulation of δ_0. Voiculescu defined $\delta_0(X)$ by

$$\delta_0(X) = n + \limsup_{\epsilon \to 0} \frac{\chi(x_1 + \epsilon s_1, \ldots, x_n + \epsilon s_n : s_1, \ldots, s_n)}{|\log \epsilon|}$$

where $s_1, \ldots, s_n$ is a semicircular family free with respect to X. One can actually also define $\delta_0(X)$ in terms of ϵ metric packings. One associates to each $\epsilon > 0$ an asymptotic ϵ packing number $\mathbb{P}_\epsilon(X)$ and ϵ covering number $\mathbb{K}_\epsilon(X)$. These definitions also make use of cutoff constants. Let's recall the definitions. For any metric space Ω and $\epsilon > 0$ denote by $P_\epsilon(\Omega)$ the maximum number in a collection of mutually disjoint open ϵ balls of Ω and by $K_\epsilon(\Omega)$ the minimum number of open ϵ balls required to cover Ω. In what follows all spaces are endowed with the $|\cdot|_2$ metric. Define successively:

$$\mathbb{P}_{\epsilon,r}(X; m, \gamma) = \limsup_{k \to \infty} k^{-2} \cdot \log(P_\epsilon(\Gamma_R(X; m, k, \gamma)),$$

$$\mathbb{P}_{\epsilon,r}(X) = \inf\{\mathbb{P}_{\epsilon,R}(X; m, \gamma) : m \in \mathbb{N}, \gamma > 0\},$$

$$\mathbb{P}_\epsilon(X) = \sup_{R>0} \mathbb{P}_{\epsilon,R}(X).$$

Similarly we define $\mathbb{K}_\epsilon(X)$ by replacing all the P_ϵ above with K_ϵ. It was shown in [5] that

$$\delta_0(X) = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(X)}{|\log \epsilon|} = \limsup_{\epsilon \to 0} \frac{\mathbb{K}_\epsilon(X)}{|\log \epsilon|}$$

Now define successively:

$$\mathbb{P}_{\epsilon,\infty}(X; m, \gamma) = \limsup_{k \to \infty} k^{-2} \cdot \log(\Gamma(X; m, k, \gamma),$$

$$\mathbb{P}_{\epsilon,\infty}(X) = \inf\{\mathbb{P}_{\epsilon,\infty}(X; m, \gamma) : m \in \mathbb{N}, \gamma > 0\}.$$

Similarly we define $\mathbb{K}_{\epsilon,\infty}$. We want to show that the packing formulation for δ_0 holds when $\mathbb{P}_\epsilon(X)$ is replaced with $\mathbb{P}_{\epsilon,\infty}(X)$ and $\mathbb{K}_\epsilon(X)$ is replaced with $\mathbb{K}_{\epsilon,\infty}(X)$. Suppose throughout this section, that X is a finite set of selfadjoint elements in M and that $R \geq 1$ is a constant greater than or equal to the maximum of the operator norms of the elements of X. We need one easy lemma which is undoubtedly known, but which we will prove out for completeness:

Lemma 2.1. *For $m_0 \in \mathbb{N}$, and $\epsilon, \gamma_0 > 0$ there exists an $m \in \mathbb{N}$ and $\gamma > 0$ such that if $\xi = (\xi_1, \ldots, \xi_n) \in \Gamma(X; m, k, \gamma)$, then $|\xi - F_R(\xi)|_2 < \epsilon$ and $(F_R(\xi_{i_1}) \cdots F_R(\xi_{i_p})) \in \Gamma_R(X; m_0, k, \gamma_0)$ where $F_r : \mathbb{R} \to [-r, r]$ is the monotone function equal to the identity on $(-r, r)$ and $F_r(\xi) = (F_r(\xi_1), \ldots, F_r(\xi_n))$.*

Proof. Denote by C the maximum over all numbers of the form $|F_{i_1}(\xi_1) \cdots F_{i_p}(\xi_p)|_2 + 1$ where $1 \leq p \leq m$ and $i_1, \ldots, i_p \in \{R, \infty\}$ (this constant C is used to satisfy the second condition). Now choose $\gamma < \frac{\epsilon}{100Cmn}$. Choose $m \in \mathbb{N}$ so large that $\frac{R^m+\gamma}{(R+\gamma)^m} < \frac{\epsilon^2}{100n^2(R^4+\gamma)}$. Suppose $\xi = (\xi_1, \ldots, \xi_n) \in \Gamma(X; m_1, k, \gamma_1)$ and denote by $\lambda_{i1}, \ldots, \lambda_{ik}$ the eigenvalues of ξ_i with multiplicity. Set $B_i = \{j \in \mathbb{N} : 1 \leq j \leq k, |\lambda_{ij}| \geq R + \gamma\}$. We have:

$$\#B_i \cdot (R+\gamma)^m \leq \sum_{j \in B_i} |\lambda_{ij}|^m \leq \sum_{j=1}^k |\lambda_j|^m \leq |Tr(\xi_i^m)| < k(R^m + \gamma).$$

Consequently $\frac{\#B_i}{k} \leq \frac{R^m+\gamma}{(R+\gamma)^m}$. By the Cauchy-Schwarz inequality

$$|\xi_i - F_{R+\gamma}(\xi_i)|_2^2 \leq \frac{1}{k} \cdot \sum_{j \in B_i} |\lambda_{ij}|^2 \leq |a^2|_2 \cdot \left(\frac{\#B_i}{k}\right)^{\frac{1}{2}} < (R^4+\gamma)^{\frac{1}{2}} \cdot \sqrt{\frac{\epsilon^2}{100n^2(R^4+\gamma)}} < \frac{\epsilon}{10n}.$$

$|F_{R+\gamma}(\xi_i) - F_R(\xi)|_2 \leq \gamma$ whence it follows that $|\xi - F_R(\xi)|_2 < \epsilon$. To see that the second claim is satisfied observe that for any $1 \leq p \leq m$ and $1 \leq i_1, \ldots, i_p \leq n$, Cauchy-Schwarz again yields

$$|tr_k(\xi_{i_1} \cdots \xi_{i_p}) - tr_k(F(\xi_{i_1}) \cdots F(\xi_{i_p}))| \leq C \cdot p \cdot |\xi - F_r(\xi)|_2.$$

Because the $|\cdot|_2$ quantity on the right hand side can be made smaller than any given $\epsilon > 0$ and C and p are constants bounded from the get-go, we're done. $\square$

Lemma 2.2.

$$\begin{aligned}\delta_0(X) = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_{\epsilon,\infty}(X)}{|\log \epsilon|} &= \limsup_{\epsilon \to 0} \frac{\mathbb{K}_{\epsilon,\infty}(X)}{|\log \epsilon|} \\ &= \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(X)}{|\log \epsilon|} \\ &= \limsup_{\epsilon \to 0} \frac{K_\epsilon(X)}{|\log \epsilon|}.\end{aligned}$$

Proof. Suppose $m_0 \in \mathbb{N}$ and $\gamma_0 > 0$. There exist by Lemma 2.1 an $m \in \mathbb{N}$ and $\gamma > 0$ such that if $\xi \in \Gamma(X; m, k, \gamma)$, then $|\xi - F_R(\xi)|_2 < \epsilon$ and it can be also arranged so that $F_R(\xi) \in \Gamma_R(X; m_0, k, \gamma_0)$. It follows from this that $K_{2\epsilon}(\Gamma(X; m, k, \gamma)) \leq K_\epsilon(\Gamma_R(X; m_0, k, \gamma_0)$ whence

$$\mathbb{K}_{2\epsilon,\infty}(X) \leq \mathbb{K}_{2\epsilon}(X; m, \gamma) \leq \mathbb{K}_{\epsilon,R}(X, m_0, \gamma_0).$$

So $\mathbb{K}_{2\epsilon,\infty}(X) \leq \mathbb{K}_{\epsilon,R}(X) \leq \mathbb{K}_\epsilon(X)$. Now clearly

$$\delta_0(X) = \limsup_{\epsilon \to 0} \frac{\mathbb{K}_\epsilon(X)}{|\log \epsilon|} \leq \limsup_{\epsilon \to 0} \frac{\mathbb{K}_{2\epsilon,\infty}(X)}{|\log 2\epsilon|} \leq \limsup_{\epsilon \to 0} \frac{\mathbb{K}_\epsilon(X)}{|\log 2\epsilon|} = \delta_0(X).$$

$\mathbb{P}_{\epsilon,\infty}(X) \geq \mathbb{K}_{2\epsilon,\infty}(X) \geq \mathbb{P}_{4\epsilon,\infty}(X)$ so this completes the proof. $\square$

3. The Inequality

Throughout assume X, Y, Z and F are finite sets of selfadjoint elements in M. Assume further that X generates a hyperfinite von Neumann algebra, an assumption we will restate for emphasis in some of the corollaries.

Definition 3.1. *Suppose for each $m \in \mathbb{N}$ and $\gamma > 0$, $\langle \xi_k \rangle_{k=1}^\infty$ is a sequence such that for large enough $k, \xi_k \in \Gamma(X; m, k, \gamma)$. The set of microstates $\Xi(F; m, k, \gamma)$ for F relative to the ξ_k is*

$$\Xi(F; m, k, \gamma) = \{\eta : (\xi_k, \eta) \in \Gamma(X \cup F; m, k, \gamma)\}.$$

Define successively for $\epsilon > 0$,

$$\mathbb{K}_\epsilon(\Xi(F; m, \gamma)) = \limsup_{k \to \infty} k^{-2} \cdot \log K_\epsilon(\Xi(F; m, k, \gamma)),$$

$$\mathbb{K}_\epsilon(\Xi(F)) = \inf\{\mathbb{K}_\epsilon(F; m, \gamma) : m \in \mathbb{N}, \gamma > 0\}.$$

where the packing quantities are taken with respect to $|\cdot|_2$. In a similar fashion, we define $\mathbb{P}_\epsilon(\Xi(F))$ by replacing the K_ϵ above with P_ϵ.

Lemma 3.2. *Suppose for each $m \in \mathbb{N}$ and $\gamma > 0$ we have a sequence $\langle \xi_k \rangle_{k=1}^\infty$ satisfying $\xi_k \in \Gamma(X; m, k, \gamma)$ for sufficiently large k. Then,*

$$\delta_0(X \cup F) = \delta_0(X) + \limsup_{\epsilon \to 0} \frac{\mathbb{K}_\epsilon(\Xi(F))}{|\log \epsilon|}.$$

Proof. First we show that the left hand side is greater than or equal to the right hand side. Suppose $t > 0$ is given. By [5] and [6] there exists an $\epsilon_0 > 0$ such that for all $\epsilon_0 > \epsilon > 0$ and any $m \in \mathbb{N}, \gamma > 0, \liminf_{k \to \infty} P_\epsilon(\Gamma(X; m, k, \gamma)) > (\delta_0(X) - t)|\log 2\epsilon|$. Now suppose $m \in \mathbb{N}$ and $\gamma > 0$ are fixed. Consider the $\langle \xi_k \rangle_{k=1}^\infty$ corresponding to the m and γ. Because the von Neumann algebra generated by X is hyperfinite, by [5] I can find a set of unitaries $\langle v_{\lambda k} \rangle_{\lambda \in \Lambda_k}$ such that $\langle v_{\lambda k} \xi_k v_{\lambda k}^* \rangle_{\lambda \in \Lambda_k}$ is an ϵ separated set with respect to the $|\cdot|_2$ norm and $\liminf_{k \to} k^{-2} \cdot \log \#\Lambda_k > (\delta_0(X) - t)|\log 2\epsilon|$. For each k pick an ϵ separated subset $\langle \zeta_k \rangle_{k=1}^\infty$ of minimal cardinality for $\Xi(F; m, k, \gamma)$ (the set of microstates for F relative to ξ_k). Now it is manifest that

$$\langle (v_{\lambda k} \xi_k v_{\lambda k}^*, v_{\lambda k} \zeta_{jk} v_{\lambda k}^*) \rangle_{(\lambda, j) \in \Lambda_k \times J_k}$$

is a subset of $\Gamma(X \cup F; m, k, \gamma)$ and moreover, it is easily checked that this set is ϵ-separated with respect to the $|\cdot|_2$ norm. Hence,

$$\begin{aligned}
\mathbb{P}_{\epsilon,\infty}(X \cup F; m, \gamma) &\geq \limsup_{k \to \infty} k^{-2} \cdot \log(\#\Lambda_k \cdot \#J_k) \\
&\geq \liminf_{k \to \infty} k^{-2} \cdot \log \#\Lambda_k + \limsup_{k \to \infty} k^{-2} \cdot \log P_\epsilon(\Xi(F; m, k, \gamma) \\
&\geq (\delta_0(X) - t)|\log 2\epsilon| + \limsup_{k \to \infty} k^{-2} \cdot \log P_\epsilon(\Xi(F; m, k, \gamma)
\end{aligned}$$

so that for $\epsilon_0 > \epsilon > 0$

$$\mathbb{P}_{\epsilon,\infty}(X \cup F) \geq (\delta_0(X) - t)|\log 2\epsilon| + \mathbb{P}_\epsilon(\Xi(F))$$

Using the packing formulation of δ_0 in [6], the fact that for any metric space $\Omega, P_\epsilon(\Omega) \geq K_{2\epsilon}(\Omega) \geq P_{4\epsilon}(\Omega)$, and dividing by $|\log \epsilon|$ and taking $\limsup_{\epsilon \to 0}$ on both sides gives

$$\begin{aligned}\delta_0(X \cup F) = \limsup_{\epsilon \to 0} \frac{\mathbb{P}_{\epsilon,\infty}(X \cup F)}{|\log \epsilon|} &\geq \limsup_{\epsilon \to 0} \frac{(\delta_0(X) - t)|\log 2\epsilon| + \mathbb{P}_\epsilon(\Xi(F))}{|\log \epsilon|} \\ &= \delta_0(X) - t + \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(\Xi(F))}{|\log \epsilon|} \\ &= \delta_0(X) - t + \limsup_{\epsilon \to 0} \frac{\mathbb{K}_\epsilon(\Xi(F))}{|\log \epsilon|}.\end{aligned}$$

$t > 0$ being arbitrary we have

$$\delta_0(X \cup F) \geq \delta_0(X) + \limsup_{\epsilon \to 0} \frac{\mathbb{P}_\epsilon(\Xi(F))}{|\log \epsilon|}.$$

For the reverse inequality by [6] there are $C, \epsilon_0 > 0$ such that for $\epsilon_0 > \epsilon > 0$ and for any $k \in \mathbb{N}$ and tractable subgroup H of U_k (in the sense of [4]) there exists an ϵ-net for U_k/H with respect to the quotient metric induced by $|\cdot|_\infty$ with cardinality no greater than $\left(\frac{C}{\epsilon}\right)^{\dim(U_k/H)}$. Write R for the maximum of the operator norms of the elements in X. Suppose $m \in \mathbb{N}$ and $\gamma > 0$. Observe that there exists $\epsilon > r > 0$ so small that if $(\xi, \eta) \in \Gamma_R(X \cup F; m, k, \gamma/2)$ and $|(\xi, \eta) - (x, a)|_2 < r$, then $(x, a) \in \Gamma(X \cup F; m, k, \gamma)$. There also exist $m_1 \in \mathbb{N}$ and $\gamma_1 > 0$ such that if $\xi, x \in \Gamma(X; m_1, k, \gamma_1)$, then there exists a $u \in U_k$ satisfying $|u\xi u^* - x|_2 < r$. Set $m_2 = m + m_1$ and $\gamma_2 = \min\{\gamma/2, \gamma_1\}$.

By [6] I can find a sequence $\langle \xi_k \rangle_{k=1}^\infty$ such that for sufficiently large k $\xi_k \in \Gamma_R(X; m_1, k, \gamma_1)$ and $\dim \xi_k' \geq k^2(1 - \delta_0(X))$. Consider the associated $\Xi(F; m, k, \gamma)$ and for each k find an ϵ-net $\langle \eta_{jk} \rangle_{j \in J_k}$ for $\Xi(F; m, k, \gamma)$ with respect to $|\cdot|_2$ of minimum cardinality. Define H_k to be the unitary group of ξ_k'. I can find for each k large enough a set of unitaries $\langle u_{gk} \rangle_{g \in G_k}$ such that their images in U_k/H_k is an ϵ-net with respect to the quotient metric induced by $|\cdot|_\infty$ and such that

$$\#\Lambda_k \leq \left(\frac{C}{\epsilon}\right)^{\delta_0(X)k^2}.$$

Consider

$$\langle (u_{gk}\xi_k u_{gk}^*, u_{gk}\eta_{jk}u_{gk}^*) \rangle_{(g,j) \in G_k \times J_k}.$$

I claim that this set is a $5\epsilon R$-net for $\Gamma(X \cup F; m_2, k, \gamma_2)$.

To see this suppose $(\xi, \eta) \in \Gamma(X \cup F; m_2, k, \gamma_2)$. By the selection of m_1 and γ_1 there exists a $u \in U_k$ such that $|u^*\xi_k u - \xi|_2 < r$. Taking into account the stipulation on r this implies that $(u^*\xi_k u, \eta) \in \Gamma(X; m, k, \gamma) \iff (\xi_k, u\eta u^*) \in \Gamma(X \cup F; m, k, \gamma)$, whence $u\eta u^* \in \Xi(F; m, k, \gamma)$. There exists an $g \in G_k$ and an $h \in H_k$ such that $|u - u_{gk}h|_\infty < \epsilon$. Consequently,

$$|u_{gk}\xi_k u_{gk}^* - \xi|_2 = |u_{gk}h\xi_k h^* u_{gk} - \xi|_2 \leq 2\epsilon R + |u\xi_k u^* - \xi|_2 \leq 3\epsilon R.$$

Now $u\eta u^* \in \Xi(F; m, k, \gamma)$ so there exists a $j \in J_k$ such that $|\eta_{jk} - u\eta u^*|_2 < \epsilon$. Because $\Xi(F; m, k, \gamma)$ is invariant under the action of H_k it follows that $h\eta_{jk}h^* \in \Xi(F; m, k, \gamma)$ and so there exists an $\ell \in J_k$ such that $|\eta_{\ell k} - h\eta_{jk}h^*|_2 < \epsilon$. So again we have

$$|u_{gk}\eta_{\ell k}u_{gk}^* - \eta|_2 < |u_{gk}h\eta_{jk}h^* u_{gk}^* - \eta|_2 + \epsilon < |u\eta_{jk}u^* - \eta|_2 + 3\epsilon R < 4\epsilon R.$$

and we have the desired claim.

It follows that

$$\begin{aligned}\mathbb{K}_{5\epsilon R,\infty}(X\cup F;m_2,\gamma_2) &\leq \limsup_{k\to\infty} k^{-2}\cdot\log(\#G_k\cdot\#J_k)\\ &\leq \log C+\delta_0(X)\cdot|\log\epsilon|+\limsup_{k\to\infty}k^{-2}\cdot\log K_\epsilon(\Xi(F;m,k,\gamma)).\end{aligned}$$

Given any $m\in\mathbb{N}$ and $\gamma>0$ we produced $m_2\in\mathbb{N}$ and $\gamma_2>0$ so that the above inequality holds for $0<\epsilon<\epsilon_0$. Thus

$$\mathbb{K}_{5\epsilon,\infty}(X\cup F)\leq\log C+\delta_0(X)\cdot|\log\epsilon|+\mathbb{K}_\epsilon(\Xi(F))).$$

Taking $\limsup_{\epsilon\to 0}$ on both sides and again using the packing formulation of δ_0 in [6] as well as Lemma 2.2. we have

$$\begin{aligned}\delta_0(X\cup F)=\limsup_{\epsilon\to0}\frac{\mathbb{K}_{5\epsilon R,\infty}(X\cup F)}{|\log 5\epsilon R|} &\leq \limsup_{\epsilon\to0}\frac{\log C+\delta_0(X)\cdot|\log\epsilon|+\mathbb{K}_\epsilon(\Xi(F))}{|\log 5\epsilon R|}\\ &= \delta_0(X)+\limsup_{\epsilon\to0}\frac{\mathbb{K}_\epsilon(\Xi(F))}{|\log\epsilon|}\end{aligned}$$

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Hyperfinite Inequality for δ_0. *If X'' is hyperfinite, then*

$$\delta_0(X\cup Y\cup Z)\leq\delta_0(X\cup Y)+\delta_0(X\cup Z)-\delta_0(X).$$

Proof. X has finite dimensional approximants, so for each $m\in\mathbb{N}$ and $\gamma>0$ we can find sequences $\langle\xi\rangle_{k=1}^\infty$ satisfying the conditions of Lemma 2.2 and consider all relative microstates with respect to these fixed sequences. For each k, $\Xi(Y\cup Z;m,k,\gamma)\subset\Xi(Y;m,k,\gamma)\times\Xi(Z;m,k,\gamma)$ so that

$$K_{2\epsilon}(\Xi(Y\cup Z;m,k,\gamma))\leq K_\epsilon(\Xi(Y;m,k,\gamma))\cdot K_\epsilon(\Xi(Z;m,k,\gamma)).$$

It follows that

$$\limsup_{\epsilon\to0}\frac{\mathbb{K}_\epsilon(\Xi(Y\cup Z))}{|\log\epsilon|}\leq\limsup_{\epsilon\to0}\frac{\mathbb{K}_\epsilon(\Xi(Y))}{|\log\epsilon|}+\limsup_{\epsilon\to0}\frac{\mathbb{K}_\epsilon(\Xi(Z))}{|\log\epsilon|}.$$

By the preceding lemma and the inequality above

$$\begin{aligned}\delta_0(X\cup Y\cup Z) &= \delta_0(X)+\limsup_{\epsilon\to0}\frac{\mathbb{K}_\epsilon(\Xi(Y\cup Z))}{|\log\epsilon|}\\ &\leq \delta_0(X)+\limsup_{\epsilon\to0}\frac{\mathbb{K}_\epsilon(\Xi(Y))}{|\log\epsilon|}+\limsup_{\epsilon\to0}\frac{\mathbb{K}_\epsilon(\Xi(Z))}{|\log\epsilon|}\\ &= \delta_0(X\cup Y)+\delta_0(X\cup Z)-\delta_0(X).\end{aligned}$$

∎

Remark 3.3. *The hyperfinite assumption on X'' is necessary. To see this consider the group inclusion $\mathbb{F}_3\subset\mathbb{F}_2\subset\mathbb{F}_3$ where $\mathbb{F}_n$ is the free group on n generators. On the von Neumann algebra level this translates to $L(\mathbb{F}_3)\simeq M_1\subset L(\mathbb{F}_2)\simeq M_2\subset L(\mathbb{F}_3)\simeq M_3$. Take X,Y and Z to be the canonical sets of freely independent semicirculars associated to M_1,M_2, and M_3, respectively. Then $\delta_0(X\cup Y)+\delta_0(X\cup Z)-\delta_0(X)=3+2-3=2<3=\delta_0(X\cup Y\cup Z)$.*

4. Five Corollaries

In this section X, Y, and Z are again finite sets of selfadjoint elements in M. Here are some corollaries of the hyperfinite inequality for δ_0:

Corollary 4.1. *Suppose X'' is hyperfinite. Assume one of the following holds:*
- $Z \subset X''$.
- *X'' is diffuse, $\delta_0(X \cup Z) \leq 1$, and $Z \subset (X \cup Y)''$.*

Then $\delta_0(X \cup Y) = \delta_0(X \cup Y \cup Z)$.

Proof. In either of the two cases Z is contained in the von Neumann algebra generated by X and Y so by [10] $\delta_0(X \cup Y) \leq \delta_0(X \cup Y \cup Z)$. For the reverse inequality observe that either situations imply $\delta_0(X \cup Z) = \delta_0(X)$. This follows in the first case from invariance of δ_0 for hyperfinite von Neumann algebras ([5]). In the second case we have by assumption and hyperfinite monotonicity that $1 \geq \delta_0(X \cup Z) \geq \delta_0(X) \geq 1$. In either cases $\delta_0(X \cup Z) = \delta_0(X)$ so by the hyperfinite inequality,

$$\delta_0(X \cup Y \cup Z) \leq \delta_0(X \cup Y) + \delta_0(X \cup Z) - \delta_0(X) = \delta_0(X \cup Y).$$

Thus, $\delta_0(X \cup Y) = \delta_0(X \cup Y \cup Z)$. □

Corollary 4.2. *If $X'' = Z''$ is hyperfinite, then $\delta_0(X \cup Y) = \delta_0(Y \cup Z)$.*

Corollary 4.3. *If $y = y^*$ lies in the center of the von Neumann algebra generated by Y, then*

$$\delta_0(Y \cup \{y\}) = \delta_0(Y).$$

Proof. Again by [10] $\delta_0(Y) \leq \delta_0(Y \cup \{y\})$. For the reverse inequality set $\alpha = \sup\{\delta_0(x) : x = x^* \in Y''\}$ (actually the supremum is achieved but I won't need that). Suppose $\epsilon > 0$. Find $x = x^* \in Y''$ such that $\alpha - \epsilon < \delta_0(x)$. Take a sequence $\langle x_k \rangle_{k=1}^{\infty}$ such that for each k, $x_k = x_k^*$ lies in the $*$-algebra generated by Y and such that $x_k \rightarrow x$ strongly. Now for every k there exists an $a_k = a_k^*$ such that the von Neumann algebra generated by a_k is equal to the von Neumann algebra generated by x_k and y and thus $\delta_0(a_k) = \delta_0(x_k, y)$. Using the fact that δ_0 is an algebraic invariant we have by the hyperfinite inequality for δ_0

$$\begin{aligned}\delta_0(Y \cup \{y\}) = \delta_0(\{x_k\} \cup Y \cup \{y\}) &\leq \delta_0(\{x_k\} \cup Y) + \delta_0(x_k, y) - \delta_0(x_k) \\ &= \delta_0(Y) + \delta_0(a_k) - \delta_0(x_k) \\ &\leq \delta_0(Y) + \alpha - \delta_0(x_k).\end{aligned}$$

Forcing $k \rightarrow \infty$ and using the fact that $\liminf_{k \rightarrow \infty} \delta_0(x_k) \geq \delta_0(x)$ (by [8]) we have that $\delta_0(Y \cup \{y\}) \leq \delta_0(Y) + \alpha - (\alpha - \epsilon) = \delta_0(Y) + \epsilon$. $\epsilon > 0$ being arbitrary, $\delta_0(Y \cup \{y\}) \leq \delta_0(Y)$. Thus, $\delta_0(Y \cup \{y\}) = \delta_0(Y)$. □

Corollary 4.4. *Suppose $x = x^* \in M, \delta_0(x, Y) = \alpha, \delta_0(Z) = \beta, \{x\} \cup Y \subset Z''$, and $Z = \{z_1, \ldots, z_n\}$. Then*

$$\beta - \alpha + n \cdot \delta_0(x) \leq \sum_{j=1}^{n} \delta_0(x, z_j).$$

Thus if $n < \beta - \alpha + n \cdot \delta_0(x)$, then for some $j, 1 < \delta_0(x, y_j)$. In particular, if Z consists of $2 \leq \beta \in \mathbb{N}$ freely independent semicircular elements, $Z = \{s_1, \ldots, s_\beta\}$ and x is any self-adjoint element in Z'' with no atoms, then for some $1 \leq j \leq \beta, 1 < \delta_0(x, s_j)$.

Proof. $\{x\} \cup Y \subset Z''$ so by [9] and the hyperfinite inequality

$$\beta = \delta_0(Z) \leq \delta_0(\{x\} \cup Y \cup \{z_1, \ldots, z_n\}) \leq \delta_0(\{x\} \cup Y \cup \{z_1, \ldots, z_{n-1}\}) + \delta_0(x, z_n) - \delta_0(x).$$

Repeating this n times we arrive at

$$\beta \leq \delta_0(\{x\} \cup Y) + \sum_{j=1}^{n} \delta_0(x, z_j) - n \cdot \delta_0(x) = \alpha - n \cdot \delta_0(x) + \sum_{j=1}^{n} \delta_0(x, z_j),$$

whence $\beta - \alpha + n \cdot \delta_0(x) \leq \sum_{j=1}^{n} \delta_0(x, z_j)$. Everything else is obvious. □

Remark 4.5. *Recall from [4] that for a finite set of selfadjoint elements F in M, if $\delta_0(F) > 1$, then the von Neumann algebra generated by F cannot be generated by a sequence of Haar unitaries $\langle u_j \rangle_{j=1}^{s}$ satisfying the condition $u_{j+1} u_j u_{j+1}^* \in \{u_1, \ldots, u_j\}''$. In particular F'' is prime and has no Cartan subalgebras. Thus, in the context of Corollary 4.4 for some j, $\{x, s_j\}''$ is prime and has no Cartan subalgebra.*

We conclude with a microstates analogue of a property of the Connes-Shlyakhtenko dimension $\triangle$:

Corollary 4.6. *If $X = \{x, x_1, \ldots, x_n\}$, $xx_i = x_i x$ for all $1 \leq i \leq n$, and the spectrum of x is diffuse, then $\delta_0(X) \leq 1$. If in addition, X has finite dimensional approximants, then $\delta_0(X) = 1$. Consequently, if F is a finite set of selfadjoint elements in M which has finite dimensional approximants and such that the von Neumann algebra generated by F has diffuse center, then $\delta_0(F) = 1$.*

Proof. By the hyperfinite inequality for δ_0, the diffuseness of x, and [9]

$$\delta_0(X) \leq \delta_0(x, \ldots, x_{n-1}) + \delta_0(x, x_n) - \delta_0(x) \leq \delta_0(x, \ldots, x_{n-1}) + 1 - 1 = \delta_0(x, x_1, \ldots, x_{n-1}).$$

Continuing inductively we have $\delta_0(X) \leq \delta_0(x) = 1$ as promised. If X has finite dimensional approximants, then by [5] $\delta_0(X) \geq \delta_0(x) = 1$ and consequently, $\delta_0(X) = 1$. The claim concerning F is immediate.

□

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